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Coherence and radiative energy transfer for linear surface gravity waves in water of constant depth

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Abstract. The radiative energy transfer for partially coherent water waves is analysed. For linear gravity waves in water of constant depth, the coherence properties are described within the framework of optical coherence theory. Exact energy transfer relations are derived and compared to similar relations for acoustical compression waves. On the basis of previous results for non-dispersive waves, an exact, geometrical description of the radiative energy transfer for dispersive water waves is derived. The theory reduces to the classical theory of radiative energy transfer within a quasihomogeneous wave model. The results are discussed in relation to the applications of the classical description for local wave energy prediction.

1. Introduction

The description of radiative energy transfer has traditionally been based on a purely phenomenological foundation: assuming that the wave energy transfer occurs along the rays of geometrical optics one describes it in terms of a radiation balance equation which totally neglects the wave nature of the radiation. However, in spite of its obvious limitations, the classical theory appears to give adequate and accurate predictions of the energy transfer for a large class of wave phenomena. For example: the classical description is extensively used in optics and astronomy for the description of energy transfer of electromagnetic waves (radiometry, radiative transfer theory) [1, 2], and essentially the same approach is used for wave prediction (hindcast models) and the description of surface wave interactions in oceanography [3–5].

Much attention has been given to the problem of providing a wave theoretical justification for the classical theory (cf [6–11] and references cited therein). For non-dispersive waves, this problem has been solved for the simplified case of free-space propagation [8–11]. The solution consists of *deriving* the classical radiative transfer equations from an exact, geometrical theory of freely propagated fields based on the wave theory of partially coherent fields by imposing well-defined approximations on the exact, geometrical theory. The classical theory is then shown to apply within a quasihomogeneous approximation for which the spatial coherence of the field is related to the classical specific intensity by a plane wave expansion that corresponds to a modified Debye integral [11, 12–14]. A similar expansion is often used for deriving the radiation transfer equation from the Bethe-Salpeter equation which describes the propagation of waves in random media [15, 16], however, that derivation is not based on a general solution of the field equations but rather on an *ad hoc* assumption of the functional form of the solution.

In the present paper, we extend the previous exact solution for non-dispersive, scalar waves [9-11] to the case of freely propagated, dispersive water waves and, to that end, analyse the radiative energy transfer for partially coherent, linear surface waves in water of constant depth. In section 2, we review the classical radiative energy transfer relations and show that they are based on a phenomenological foundation which neglects the wave nature of the radiation. In section 3, we develop the coherence theory description of linear surface gravity waves and, in section 4, we apply this theory to derive the exact radiative energy transfer relations for such wavefields. In section 5, we present an exact, geometrical description in terms of radiative energy transfer equations which are almost identical to the classical ones and, in section 6, we show that this description reduces to the classical radiative transfer relations within an approximate quasihomogeneous wave model. Finally, in section 7, we summarize the results of the paper and briefly discuss their relevance to the problem of wave energy prediction.

2. The classical theory of radiative energy transfer

The classical description of radiative energy transfer dates back to the works of Kirchhoff and Planck (cf [1]) which form the basis of the radiative transfer theory of Chandrasekhar [2]. The main assumption of this description is that the radiative energy transfer is described by one single scalar function—the 'specific intensity' or 'radiance' function $I_0(\boldsymbol{x}, \boldsymbol{s})$—which describes the local distribution of the spectral energy flux density over ray directions $\boldsymbol{s}$ ($\boldsymbol{s}$ being a unit direction vector). This specific intensity function is assumed to possess a number of properties which are postulated on the basis of geometrical optics.

According to this description, the mean energy surface density $E(\boldsymbol{x})$ and the mean energy flux surface density $\boldsymbol{F}(\boldsymbol{x})$ at a point $\boldsymbol{x} = \{x, y\}$ in the surface plane can be represented as integrals over corresponding spectral densities, i.e.

$$E(\boldsymbol{x}) = \int_0^\infty E(\boldsymbol{x}, \omega)\,\mathrm{d}\omega \tag{2.1}$$

$$\boldsymbol{F}(\boldsymbol{x}) = \int_0^\infty \boldsymbol{F}(\boldsymbol{x}, \omega)\,\mathrm{d}\omega \tag{2.2}$$

where ω is the angular frequency, and the spectral densities $E(\boldsymbol{x}, \omega)$ and $\boldsymbol{F}(\boldsymbol{x}, \omega)$ are given in terms of the specific intensity $I_0(\boldsymbol{x}, \boldsymbol{s})$ by:

$$E(\boldsymbol{x}, \omega) = \frac{1}{v_g} \int_0^{2\pi} I_0(\boldsymbol{x}, \boldsymbol{s})\,\mathrm{d}\theta(\boldsymbol{s}) \tag{2.3}$$

$$\boldsymbol{F}(\boldsymbol{x}, \omega) = \int_0^{2\pi} I_0(\boldsymbol{x}, \boldsymbol{s})\boldsymbol{s}\,\mathrm{d}\theta(\boldsymbol{s}). \tag{2.4}$$

Here, $v_g = \mathrm{d}\omega/\mathrm{d}k$ is the group velocity, where $k = 2\pi/\lambda$ is the wavenumber and λ is the wavelength at angular frequency ω, $\boldsymbol{s} = \{s_x, s_y\} = \{\cos\theta(\boldsymbol{s}), \sin\theta(\boldsymbol{s})\}$ is a unit direction vector, and $\mathrm{d}\theta(\boldsymbol{s})$ is the infinitesimal angular element around the direction vector $\boldsymbol{s}$. Equations (2.3) and (2.4) imply the assumption that the radiative energy is propagated along geometrical rays so that the flux density associated with the ray direction $\boldsymbol{s}$ is equal to the corresponding energy density times the group velocity. The resulting energy and energy flux densities are then given by direct summation over all the contributing ray directions.

The specific intensity function $I_0(x, s)$ is assumed to satisfy the geometrical propagation law

$$s \cdot \nabla I_0(x, s) = 0 \tag{2.5}$$

so that, for each s, $I_0(x, s)$ is constant along rays in the s direction. As an energy flux density, I_0 is also assumed to be positive, i.e.

$$I_0(x, s) \geqslant 0. \tag{2.6}$$

Equation (2.5) applies for the case of free propagation which is the only case that will be considered here. If absorption or scattering is present, the radiation balance equation (2.5) must be modified to include a source term for appropriate modelling of such phenomena [2]. Note that equations (2.2), (2.4) and (2.5) imply that F satisfies the energy conservation equation for stationary fields:

$$\nabla \cdot F(x) = 0. \tag{2.7}$$

As pointed out already by Planck [1], the classical description, as summarized in equations (2.1)–(2.6), is a *phenomenological* theory which neglects the wave nature of the radiation. Although this description is *plausible* from a geometrical optics point of view, it is not *a priori* obvious that the predictions of this theory actually correspond to the energy transfer of the wavefield. To obtain a rational foundation of the classical description, we must *derive* it on the basis of statistical wave theory and show that it is in agreement with the physics of the wave phenomenon considered. For non-dispersive, scalar waves such a derivation has been given in [9–11]. There, it is shown that the classical theory applies within an approximate, quasihomogeneous wave model. Here, we will give a similar derivation for partially coherent, linear, dispersive water waves.

3. Coherence theory for linear water waves

In this section, we first review the basic relations for surface gravity waves in water of constant depth. We then develop a statistical description for such waves and show that their average properties can be described within the mathematical framework of optical coherence theory.

In the linear approximation, surface gravity waves in water of constant depth d are described by the following set of equations [17]:

$$\nabla^2 \phi = 0 \qquad -d \leqslant z \leqslant 0 \tag{3.1}$$

$$\left. \begin{aligned} & g\eta + \frac{\partial \phi}{\partial t} = 0 \\ & \frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \end{aligned} \right\} \qquad z = 0 \tag{3.2}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad z = -d. \tag{3.3}$$

Here, $\phi=\phi(\boldsymbol{r},t)$ (with $\boldsymbol{r}=\{x,y,z\}$) is the velocity potential (i.e., $\boldsymbol{v}(\boldsymbol{r},t)=\nabla\phi(\boldsymbol{r},t)$ is the fluid velocity), g is the acceleration of gravity, and $\eta=\eta(\boldsymbol{x},t)$ (with $\boldsymbol{x}=\{x,y\}$) is the surface elevation due to the wave motion. The free surface is at $z=0$ and the horizontal bottom is at $z=-d$. Equations (3.1)–(3.3) describe the case with no viscosity and friction losses, but in the linear approximation the wave heights and particle velocities are so small that viscosity and friction losses at the bottom are negligible. If friction losses are appreciable the amplitudes will generally be so high that non-linearities must also be taken into account.

In this linear approximation the energy transport is determined only by the travelling-wave solutions of equations (3.1)–(3.3). For travelling waves, we may represent the real functions ϕ and η in terms of their complex analytical signals Φ and V which, obviously, also satisfy equations (3.1)–(3.3), i.e.

$$\phi=\operatorname{Re}\Phi \qquad \eta=\operatorname{Re}V. \tag{3.4}$$

It is now easily verified that the general travelling-wave solutions of V and Φ can be expressed as [17–19]

$$V(\boldsymbol{x},t)=\int_0^\infty U(\boldsymbol{x},\omega)\exp(-\mathrm{i}\omega t)\,\mathrm{d}\omega \tag{3.5}$$

and

$$\Phi(\boldsymbol{r},t)=\int_0^\infty \frac{\cosh\{k(z+d)\}}{\cosh(kd)}\frac{g}{\mathrm{i}\omega}U(\boldsymbol{x},\omega)\exp(-\mathrm{i}\omega t)\,\mathrm{d}\omega \tag{3.6}$$

where $U(\boldsymbol{x},\omega)$ is the spectral amplitude of the surface elevation, i.e.

$$U(\boldsymbol{x},\omega)=\frac{1}{\pi}\int_{-\infty}^\infty \eta(\boldsymbol{x},t)\exp(\mathrm{i}\omega t)\,\mathrm{d}t \qquad \omega>0 \tag{3.7}$$

ω is the angular frequency, and $k=k(\omega)$ is determined by the positive root of the dispersion relation [3, 4, 17]

$$\omega^2=gk\tanh(kd). \tag{3.8}$$

Substituting equations (3.4)–(3.6) into equations (3.1)–(3.3), we see that $U(\boldsymbol{x},\omega)$ satisfies the Helmholtz equation [18, 19]

$$\nabla^2U(\boldsymbol{x},\omega)+k^2U(\boldsymbol{x},\omega)=0. \tag{3.9}$$

Thus, by a spectral decomposition, the propagation of linear surface gravity waves in water of constant depth can be described by the same reduced wave equation (the Helmholtz equation) that is used for describing other, non-dispersive wavefields.

For a fluctuating, partially coherent wavefield, we can only predict statistical moments of the potential $\phi(\boldsymbol{r},t)$ and the surface elevation $\eta(\boldsymbol{x},t)$. If the wavefield obeys *stationary* statistics, so that averages of the type $\langle\phi(\boldsymbol{r}_1,t_1)\phi(\boldsymbol{r}_2,t_2)\rangle$ depend on t_1 and t_2 only through the time delay $\tau=t_2-t_1$, then all second-order statistical averages can be expressed in terms of complex coherence functions. Since the complex analytical signals are band limited (with only positive frequencies), stationary, random functions of time, the following rule applies for the evaluation of the averages:

$$\langle\phi(\boldsymbol{r}_1,t_1)\phi(\boldsymbol{r}_2,t_2)\rangle=\langle\operatorname{Re}[\Phi(\boldsymbol{r}_1,t_1)]\operatorname{Re}[\Phi(\boldsymbol{r}_2,t_2)]\rangle=\tfrac{1}{2}\operatorname{Re}\langle\Phi^*(\boldsymbol{r}_1,t_1)\Phi(\boldsymbol{r}_2,t_2)\rangle \tag{3.10}$$

where the asterisk denotes complex conjugation and the brackets denote statistical averages, i.e. ensemble averages. Here, we have made use of the relation

$$\mathrm{Re}\, z_1\, \mathrm{Re}\, z_2 = \tfrac{1}{4}(z_1+z_1^*)(z_2+z_2^*) = \tfrac{1}{4}(z_1^* z_2 + z_1 z_2^* + z_1 z_2 + z_1^* z_2^*) = \tfrac{1}{2}\, \mathrm{Re}(z_1^* z_2 + z_1 z_2) \quad (3.11)$$

and the fact that $\langle \Phi(\boldsymbol{r}_1, t_1)\Phi(\boldsymbol{r}_2, t_2)\rangle = 0$ for a stationary, random process [20]. For our analysis, we need the following coherence functions:

$$\Gamma_\Phi(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \langle \Phi^*(\boldsymbol{r}_1, t-\tau)\Phi(\boldsymbol{r}_2, t)\rangle \quad (3.12)$$

and

$$\Gamma_v(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \langle V^*(\boldsymbol{x}_1, t-\tau)\, V(\boldsymbol{x}_2, t)\rangle. \quad (3.13)$$

From equations (3.5) and (3.6), it follows that Γ_Φ and Γ_v are both determined by the following cross-spectral coherence function

$$\Gamma_u(\boldsymbol{x}_1, \boldsymbol{x}_2, \omega_1, \omega_2) = \langle U^*(\boldsymbol{x}_1, \omega_1)\, U(\boldsymbol{x}_2, \omega_2)\rangle. \quad (3.14)$$

Substituting equation (3.5) in equation (3.13), we see from the stationarity condition that the cross-spectral coherence function must satisfy

$$\Gamma_u(\boldsymbol{x}_1, \boldsymbol{x}_2, \omega', \omega) = W_u(\boldsymbol{x}_1, \boldsymbol{x}_2, \omega)\delta(\omega-\omega') \quad (3.15)$$

where

$$W_u(\boldsymbol{x}_1, \boldsymbol{x}_2, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma_v(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau)\exp(\mathrm{i}\omega\tau)\, \mathrm{d}\tau \quad (3.16)$$

is the cross-spectral density function (cf [6] and [21]) for the surface elevation.

By multiplying equation (3.9) and its complex conjugate by, respectively, $U^*(\boldsymbol{x}', \omega')$ and $U(\boldsymbol{x}', \omega')$, we obtain from equation (3.15) after averaging and integration over ω'

$$\nabla_1^2 W_u(\boldsymbol{x}_1, \boldsymbol{x}_2, \omega) + k^2 W_u(\boldsymbol{x}_1, \boldsymbol{x}_2, \omega) = 0 \quad (3.17a)$$

$$\nabla_2^2 W_u(\boldsymbol{x}_1, \boldsymbol{x}_2, \omega) + k^2 W_u(\boldsymbol{x}_1, \boldsymbol{x}_2, \omega) = 0. \quad (3.17b)$$

where ∇_1 and ∇_2 operate, respectively, on the $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ coordinates of $W_u(\boldsymbol{x}_1, \boldsymbol{x}_2, \omega)$, and where $k = k(\omega)$ is as given by equation (3.8). Equations (3.17*a*, *b*) are identical to Wolf's reduced wave equations for the propagation of the cross-spectral density function [6, 22]. We note that the only influence of the dispersion enters through the ω dependence of k as given by equation (3.8).

Substituting equation (3.6) in equation (3.12), we obtain from equations (3.14) and (3.15)

$$\Gamma_\Phi(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \int_0^{\infty} W_\Phi(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)\exp(-\mathrm{i}\omega\tau)\, \mathrm{d}\omega \quad (3.18)$$

where the cross-spectral density function W_Φ for the velocity potential is given by

$$W_\Phi(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = \frac{g^2}{\omega^2}\frac{\cosh[k(z_1+d)]\cosh[k(z_2+d)]}{\cosh^2(kd)} W_u(\boldsymbol{x}_1, \boldsymbol{x}_2, \omega). \quad (3.19)$$

Here, $\boldsymbol{r}_j = \boldsymbol{x}_j + z_j\boldsymbol{e}_z$ $(j = 1, 2)$ where $\boldsymbol{e}_z$ is the unit vector in the z-direction.

Equations (3.18), (3.19) and the inverse transform of equation (3.16), i.e.

$$\Gamma_v(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau) = \int_0^{\infty} W_u(\boldsymbol{x}_1, \boldsymbol{x}_2, \omega)\exp(-\mathrm{i}\omega\tau)\, \mathrm{d}\omega \quad (3.20)$$

provide a *complete* description of the second-order moments of the wavefield in terms of the cross-spectral density function of the surface elevation W_u. Equations (3.18)-(3.20) and the propagation equations (3.17*a*, *b*) constitute the coherence theory description of fluctuating, linear surface gravity waves in water of constant depth.

For the description of radiative energy transfer, it is useful to express the cross-spectral density function as a function of mean and difference coordinates, i.e. we use the function $W(\boldsymbol{x}, \boldsymbol{\xi}) = W_u(\boldsymbol{x}-\boldsymbol{\xi}/2, \boldsymbol{x}+\boldsymbol{\xi}/2, \omega)$ where the new coordinates are $\boldsymbol{x} = (\boldsymbol{x}_1+\boldsymbol{x}_2)/2$ and $\boldsymbol{\xi} = \boldsymbol{x}_2 - \boldsymbol{x}_1$. In terms of the new coordinates, the propagation equations (3.17*a*, *b*) are transformed into Dolin's equations [23]:

$$\nabla_\xi^2 W(\boldsymbol{x}, \boldsymbol{\xi}) + k^2 W(\boldsymbol{x}, \boldsymbol{\xi}) + \tfrac{1}{4}\nabla^2 W(\boldsymbol{x}, \boldsymbol{\xi}) = 0 \tag{3.21a}$$

$$\nabla_\xi \cdot \nabla W(\boldsymbol{x}, \boldsymbol{\xi}) = 0 \tag{3.21b}$$

where ∇ and ∇_ξ operate, respectively, on the $\boldsymbol{x}$ and $\boldsymbol{\xi}$ coordinates. We have now arrived at precisely the same formalism for the description of partially coherent water waves which was used in [9-11] for the description of non-dispersive scalar waves. However, before we make further use of this analogy, we must consider the exact energy relations for partially coherent water waves.

4. Wave theory of radiative energy transfer

Here, we first review the basic energy transfer relations for linear waves in water of constant depth. We then make use of the results of the preceding section and derive exact radiative transfer relations for partially coherent waves.

The energy surface density of the wavefield is obtained by considering the kinetic and potential energy density of the fluid motion associated with the wave propagation, i.e. [3]

$$\begin{aligned}\mathscr{E}(\boldsymbol{x}, t) &= \rho g \int_0^\eta z\,\mathrm{d}z + \tfrac{1}{2}\rho \int_{-d}^0 (\nabla\phi)^2\,\mathrm{d}z \\ &= \tfrac{1}{2}\rho\left\{g\eta^2 + \int_{-d}^0 (\nabla\phi)^2\,\mathrm{d}z\right\}\end{aligned} \tag{4.1}$$

where ρ is the mass density of the fluid. The energy flux surface density $\mathscr{F}$ is defined by the energy conservation equation [3]

$$\frac{\partial \mathscr{E}}{\partial t} + \nabla \cdot \mathscr{F} = 0. \tag{4.2}$$

It can be shown from equations (3.1)-(3.3) that equation (4.2) is satisfied if [3]

$$\mathscr{F}(\boldsymbol{x}, t) = -\rho \int_{-d}^0 \frac{\partial\phi}{\partial t}\nabla\phi\,\mathrm{d}z. \tag{4.3}$$

In principle one may add any vector with zero divergence to the expression in equation (4.3) and still have equation (4.2) satisfied. However, that ambiguity is removed by the additional requirement that the flux density be a quadratic functional in field variables which approaches zero as the wave motion vanishes. Equations (4.1)-(4.3) describe the instantaneous energy relations. For a partially coherent wavefield, instantaneous quantities cannot be predicted. Instead, we must then consider average quantities.

The average energy relations are obtained by averaging equations (4.1) and (4.3). By straightforward calculations, we find by using equations (3.10), (3.12), (3.13), (3.18), and (3.20):

$$E(x) = \langle \mathscr{E}(x, t) \rangle = \int_0^\infty E(x, \omega)\, \mathrm{d}\omega \tag{4.4}$$

$$F(x) = \langle \mathscr{F}(x, t) \rangle = \int_0^\infty F(x, \omega)\, \mathrm{d}\omega \tag{4.5}$$

where

$$E(x, \omega) = \frac{\rho}{4}\left\{ g W_u(x, x, \omega) + \int_{-d}^{0} [\nabla_1 \nabla_2 W_\Phi(r_1, r_2, \omega)]_{r_1 = r_2 = r}\, \mathrm{d}z \right\} \tag{4.6}$$

$$F(x, \omega) = \frac{\rho\omega}{4\mathrm{i}} \int_{-d}^{0} [(\nabla_2 - \nabla_1) W_\Phi(r_1, r_2, \omega)]_{r_1 = r_2 = r}\, \mathrm{d}z. \tag{4.7}$$

In obtaining equation (4.7), we have made use of the symmetry relation $W_\Phi(r_2, r_1, \omega) = W_\Phi^*(r_1, r_2, \omega)$. Note, that equations (4.4) and (4.5) are equal to the corresponding equations (2.1) and (2.2) of the classical description, but instead of the phenomenological equations (2.3) and (2.4) we now have the exact expressions in equations (4.6) and (4.7) for the spectral densities.

Substituting from equation (3.19) in equations (4.6) and (4.7), we find

$$\begin{aligned} E(x, \omega) = \frac{\rho}{4}\Bigg\{ & W_u(x, x, \omega)\left[g + \left(\frac{kg}{\omega}\right)^2 \int_{-d}^{0} \frac{\sinh^2[k(z+d)]}{\cosh^2(kd)}\, \mathrm{d}z \right] \\ & + (g/\omega)^2 [\nabla_1 \nabla_2 W_u(x_1, x_2, \omega)]_{x_1 = x_2 = x} \cdot \int_{-d}^{0} \frac{\cosh^2[k(z+d)]}{\cosh^2(kd)}\, \mathrm{d}z \Bigg\} \end{aligned} \tag{4.8}$$

and

$$F(x, \omega) = \frac{\rho g^2}{4\mathrm{i}\omega} [(\nabla_2 - \nabla_1) W_u(x_1, x_2, \omega)]_{x_1 = x_2 = x} \cdot \int_{-d}^{0} \frac{\cosh^2[k(z+d)]}{\cosh^2(kd)}\, \mathrm{d}z. \tag{4.9}$$

The z-integrals in equations (4.8) and (4.9) are easily evaluated. They are

$$\begin{aligned} \int_{-d}^{0} \frac{\sinh^2[k(z+d)]}{\cosh^2(kd)}\, \mathrm{d}z &= \frac{\tanh(kd)}{2k}\left[1 - \frac{2kd}{\sinh(2kd)}\right] \\ &= \frac{1}{g}(v_p^2 - v_p v_g) \end{aligned} \tag{4.10}$$

$$\begin{aligned} \int_{-d}^{0} \frac{\cosh^2[k(z+d)]}{\cosh^2(kd)}\, \mathrm{d}z &= \frac{\tanh(kd)}{2k}\left[1 + \frac{2kd}{\sinh(2kd)}\right] \\ &= \frac{1}{g} v_p v_g. \end{aligned} \tag{4.11}$$

Here, we have made use of the dispersion relation (equation (3.8)) and the expressions for the phase velocity v_p and the group velocity v_g:

$$v_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh(kd)} \tag{4.12}$$

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{2} v_p \left[1 + \frac{2kd}{\sinh(2kd)}\right]. \tag{4.13}$$

Substituting equations (4.10) and (4.11) in equations (4.8) and (4.9), we get

$$E(x, \omega) = \frac{\rho g}{4}\left\{\left[2 - \frac{v_g}{v_p}\right] W_u(x, x, \omega) + \frac{v_g}{v_p}\frac{1}{k^2}[\nabla_1\nabla_2 W_u(x_1, x_2, \omega)]_{x_1=x_2=x}\right\} \tag{4.14}$$

$$F(x, \omega) = \frac{1}{4}\rho g v_g \frac{1}{\mathrm{i}k}[(\nabla_2 - \nabla_1) W_u(x_1, x_2, \omega)]_{x_1=x_2=x}. \tag{4.15}$$

We now change to mean and difference coordinates as in section 3. Straightforward transformation then gives

$$[(\nabla_2 - \nabla_1) W_u(x_1, x_2, \omega)]_{x_1=x_2=x} = 2[\nabla_\xi W(x, \xi)]_{\xi=0} \tag{4.16}$$

and

$$\nabla_1\nabla_2 W_u(x_1, x_2, \omega) = \tfrac{1}{4}\nabla^2 W(x, \xi) - \nabla_\xi^2 W(x, \xi). \tag{4.17}$$

Combining equation (4.17) with equation (3.21a), we find

$$[\nabla_1\nabla_2 W_u(x_1, x_2, \omega)]_{x_1=x_2=x} = \tfrac{1}{2}\nabla^2 W(x, 0) + k^2 W(x, 0) \tag{4.18}$$

since $\nabla_1 = \nabla/2 - \nabla_\xi$ and $\nabla_2 = \nabla/2 + \nabla_\xi$. When equations (4.16) and (4.18) are substituted in equations (4.14) and (4.15), we finally obtain

$$E(x, \omega) = \frac{1}{2}\rho g\left(1 + \frac{v_g}{v_p}\frac{1}{4k^2}\nabla^2\right) W(x, 0) \tag{4.19}$$

$$F(x, \omega) = \frac{1}{2}\rho g v_g \frac{1}{\mathrm{i}k}[\nabla_\xi W(x, \xi)]_{\xi=0}. \tag{4.20}$$

Equations (4.19) and (4.20) for the spectral densities, the spectral integrals (equations (4.4) and (4.5)), and the propagation relations (equations (3.21a, b)) constitute the *exact wave description of radiative energy transfer* for fluctuating, linear surface gravity waves in water of constant depth.

Equations (4.19) and (4.20) are amazingly similar to the corresponding relations for three-dimensional, non-dispersive, acoustical waves derived in [9] and [11], i.e. to

$$E(\boldsymbol{r}, \omega) = \frac{1}{2}\frac{\rho}{v_p^2}\left(1 + \frac{1}{4k^2}\nabla^2\right) W(\boldsymbol{r}, 0) \tag{4.21}$$

$$F(\boldsymbol{r}, \omega) = \frac{1}{2}\frac{\rho}{v_p}\frac{1}{\mathrm{i}k}[\nabla_\zeta W(\boldsymbol{r}, \zeta)]_{\zeta=0} \tag{4.22}$$

where ρ is the density of the medium and $W(\boldsymbol{r}, \zeta)$ is the cross-spectral density for the acoustical pressure variations expressed in the three-dimensional mean and difference coordinates $\boldsymbol{r}$ and ζ. Except for the constants, the only differences between equations (4.19), (4.20) and equations (4.21), (4.22) are that the latter apply for non-dispersive, three-dimensional waves for which $v_g = v_p$. Therefore, the same basic formulas apply in the two cases. Our results show a close analogy between both the energy relations and the propagation equations for surface gravity waves and three-dimensional acoustical waves. This analogy illustrates a basic unity of wave phenomena which are physically very different. It also demonstrates that spectral methods are extremely powerful for analysing different wave phenomena.

In concluding this section, we notice that, in general, we have no equipartition of the kinetic and potential energy density for a partially coherent wavefield. From equations (4.1) and (4.6), we find that the mean spectral potential energy density is given by

$$E_p(x, \omega) = \tfrac{1}{4}\rho g W(x, 0) \tag{4.23}$$

and from equation (4.19) it follows that the corresponding density for the kinetic energy is:

$$E_k(x, \omega) = E(x, \omega) - E_p(x, \omega) = \frac{1}{4}\rho g\left[W(x, 0) + \frac{v_g}{v_p}\frac{1}{2k^2}\nabla^2 W(x, 0)\right]. \tag{4.24}$$

Clearly, $E_k(x, \omega)$ is generally different from $E_p(x, \omega)$. A similar violation of the equipartition principle occurs also for acoustical waves. Strictly, the equipartition principle holds only for perfectly *homogeneous* wavefields, i.e. fields for which the intensity $W(x, 0)$ is constant as a function of position x so that the energy density is proportional to the intensity. However, the equipartition principle is seen to hold approximately if $W(x, 0)$ varies little with x on a scale of $1/k = \lambda/(2\pi)$.

5. Exact, geometrical radiative transfer theory

Having established that the transformed Wolf equations (3.21*a*, *b*) describe the cross-spectral density function for partially coherent water waves, and having derived the exact energy relations (equations (4.19) and (4.20)), we now use the same approach as in [11] to derive exact, geometrical energy transfer relations for water waves. To that end we first give an exact, formal solution of equations (3.21*a*, *b*) and show that it leads to an exact, geometrical description of both the radiative energy transfer and the coherence properties of the field in terms of a generalized specific intensity function $I(x, s)$ which, like the classical specific intensity $I_0(x, s)$, is constant along rays.

If we assume free propagation, i.e. neglect evanescent waves, the general solution of equations (3.17*a*, *b*) can be expanded in homogeneous plane waves as [11]:

$$W_u(x_1, x_2, \omega) = \frac{k}{2\pi}\int_0^{2\pi}\int_0^{2\pi} C(s_1, s_2)\exp[ik(s_2\cdot x_2 - s_1\cdot x_1)]\,d\theta(s_1)\,d\theta(s_2) \tag{5.1}$$

where $C(s_1, s_2)$ is the directional correlation of the plane wave components in the two directions s_1 and s_2 and $d\theta(s_j)$ is the infinitesimal angular element around the unit vector s_j $(j = 1, 2)$. Introducing the mean and difference coordinates $x, \xi, Q = (s_1 + s_2)/2 = Qs$, and $q = s_2 - s_1$, where $Q = \sqrt{1 - q^2/4}$, equation (5.1) can be written as

$$W(x, \xi) = \frac{k}{2\pi}\int_0^{2\pi}\int_0^{2\pi} S(Q, q)\exp[ik(q\cdot x + Q\cdot\xi)]\,d\theta(s_1)\,d\theta(s_2) \tag{5.2}$$

where $S(Q, q) = C(Q - q/2, Q + q/2)$. Choosing the y axis along $Q = Qs$, we easily see that $d\theta(s_1)\,d\theta(s_2) = d\theta(s)\,dq_x/Q$ where q_x is the component along the line $q\cdot s = 0$. Equation (5.2) can then be written as

$$W(x, \xi) = \int_0^{2\pi}\left\{\frac{k}{2\pi}\int \frac{S(\sqrt{1 - q^2/4}s, q)}{\sqrt{1 - q^2/4}}\,\delta(q\cdot s) \times\exp[ik(q\cdot x + \sqrt{1 - q^2/4}s\cdot\xi)]\,d^2q\right\}d\theta(s) \tag{5.3}$$

where $d^2q = dq_x\,dq_y$ and the Dirac δ-function $\delta(q\cdot s)$ in the inner integral implies that the integral has a contribution only along the line $q\cdot s = 0$, i.e. along q_x. By substitution, we easily see that equation (5.3) is a solution of equations (3.21*a*) and (3.21*b*).

Note that the inner integral in equation (5.3) is a Fourier expansion of the $\boldsymbol{x}$ dependence. Since no evanescent waves contribute in the expansion in equation (5.1) we have $S(\boldsymbol{Q}, \boldsymbol{q})=0$ for $q \geqslant 2$, and the $\boldsymbol{q}$ dependence in the roots in the denominator and the exponent in equation (5.3) can always be developed in a power series in $\boldsymbol{q}$. Any such dependence on $\mathrm{i}k\boldsymbol{q}$ in the inner integral can then be replaced by the same operator dependence on ∇ outside the inner integral, where the operator is to be developed in powers of ∇. In the following we will make extensive use of this property in order to derive the exact geometrical theory of radiative transfer.

Substituting the solution in equation (5.3) in equation (4.20), we have

$$\boldsymbol{F}(\boldsymbol{x}, \omega)=\frac{1}{2} \rho g v_{\mathrm{g}} \int_0^{2\pi}\left[\frac{k}{2\pi} \int S(\sqrt{1-q^2/4}\boldsymbol{s}, \boldsymbol{q}) \delta(\boldsymbol{q} \cdot \boldsymbol{s}) \exp(\mathrm{i}k\boldsymbol{q} \cdot \boldsymbol{x}) \, \mathrm{d}^2 q\right] \boldsymbol{s} \, \mathrm{d}\theta(\boldsymbol{s}) \tag{5.4}$$

which can be directly written as

$$\boldsymbol{F}(\boldsymbol{x}, \omega)=\int_0^{2\pi} I(\boldsymbol{x}, \boldsymbol{s}) \boldsymbol{s} \, \mathrm{d}\theta(\boldsymbol{s}) \tag{5.5}$$

where

$$I(\boldsymbol{x}, \boldsymbol{s})=\frac{1}{2} \rho g v_{\mathrm{g}} \frac{k}{2\pi} \int S(\sqrt{1-q^2/4}\boldsymbol{s}, \boldsymbol{q}) \delta(\boldsymbol{q} \cdot \boldsymbol{s}) \exp(\mathrm{i}k\boldsymbol{q} \cdot \boldsymbol{x}) \, \mathrm{d}^2 q \tag{5.6}$$

is a *generalized specific intensity* which is seen to be constant along rays, i.e. it satisfies

$$\boldsymbol{s} \cdot \nabla I(\boldsymbol{x}, \boldsymbol{s})=0. \tag{5.7}$$

This result directly follows by substituting equation (5.6) in the left-hand side of equation (5.7): the operator $\boldsymbol{s} \cdot \nabla$ gives rise to a multiplicative factor $\mathrm{i}k\boldsymbol{q} \cdot \boldsymbol{s}$ in the inner integral, and then the integral vanishes since it only has a contribution along the line $\boldsymbol{q} \cdot \boldsymbol{s}=0$.

When equation (5.3) is substituted into equation (4.19), we again make use of the fact that a dependence on $\mathrm{i}k\boldsymbol{q}$ in the roots in the inner integral can be replaced by the same operator dependence on ∇ outside the integral. It is then easily seen from equation (5.6) that

$$\begin{aligned} E(\boldsymbol{x}, \omega) &= \frac{1+\dfrac{v_{\mathrm{g}}}{v_{\mathrm{p}}} \dfrac{1}{4k^2} \nabla^2}{\sqrt{1+\dfrac{1}{4k^2} \nabla^2}} \frac{1}{v_{\mathrm{g}}} \int_0^{2\pi} I(\boldsymbol{x}, \boldsymbol{s}) \, \mathrm{d}\theta(\boldsymbol{s}) \\ &= \left[1+\left(\frac{v_{\mathrm{g}}}{v_{\mathrm{p}}}-\frac{1}{2}\right) \frac{1}{4k^2} \nabla^2+\ldots\right] \frac{1}{v_{\mathrm{g}}} \int_0^{2\pi} I(\boldsymbol{x}, \boldsymbol{s}) \, \mathrm{d}\theta(\boldsymbol{s}) \end{aligned} \tag{5.8}$$

where we have developed the operator in powers of ∇^2.

Except for the operator in equation (5.8), the exact equations (5.5), (5.7), and (5.8) are of the same form as the corresponding classical expressions (equations (2.3)–(2.5)). If $I(\boldsymbol{x}, \boldsymbol{s})$ varies little with $\boldsymbol{x}$ on a scale of $1/k=\lambda/(2\pi)$ (with λ being the wavelength), then only the leading term of the series development in equation (5.8) will contribute significantly and this equation will also be of the same form as the corresponding classical equation, i.e.

$$E(\boldsymbol{x}, \omega)=\frac{1}{v_{\mathrm{g}}} \int_0^{2\pi} I(\boldsymbol{x}, \boldsymbol{s}) \, \mathrm{d}\theta(\boldsymbol{s}). \tag{5.9}$$

Note that equation (5.9) is valid within the same approximation as the energy equipartition principle (cf section 4).

Replacing the $ik\boldsymbol{q}$ dependence in the roots in the denominator and the exponent of equation (5.3) by the same operator dependence on ∇ outside the inner integral, we can now express the cross-spectral density function in terms of the generalized specific intensity by the operator equation

$$W(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{2}{\rho g v_g} \int_0^{2\pi} \frac{1}{\sqrt{1+\nabla^2/(4k^2)}} \exp(\mathrm{i}\sqrt{k^2+\nabla^2/4}\boldsymbol{s}\cdot\boldsymbol{\xi}) I(\boldsymbol{x}, \boldsymbol{s})\, \mathrm{d}\theta(\boldsymbol{s}) \tag{5.10a}$$

where the operator is to be developed in powers of ∇^2, i.e.

$$W(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{2}{\rho g v_g} \int_0^{2\pi} \exp(\mathrm{i}k\boldsymbol{s}\cdot\boldsymbol{\xi}) \bigg[1 + \frac{1}{8}(\mathrm{i}k\boldsymbol{s}\cdot\boldsymbol{\xi} - 1)\frac{\nabla^2}{k^2} + \frac{1}{128}\{3 - 3\mathrm{i}k\boldsymbol{s}\cdot\boldsymbol{\xi} - (k\boldsymbol{s}\cdot\boldsymbol{\xi})^2\}\frac{\nabla^4}{k^4} + \ldots \bigg] I(\boldsymbol{x}, \boldsymbol{s})\, \mathrm{d}\theta(\boldsymbol{s}). \tag{5.10b}$$

In the short wavelength limit, $k \to \infty$, or if $I(\boldsymbol{x}, \boldsymbol{s})$ is a 'slow' function of $\boldsymbol{x}$, only the leading term of this expansion will contribute. To see that equation (5.10a) is equivalent with equation (5.3) we only have to substitute $I(\boldsymbol{x}, \boldsymbol{s})$ from equation (5.6) in equation (5.10a) and replace the dependence on ∇ in the operator expression by the same dependence on $ik\boldsymbol{q}$ the inner integral.

If we have no directional ambiguity, i.e. do not simultaneously have contributions from rays in the directions $\boldsymbol{s} = \{s_x, |s_y|\}$ and $\boldsymbol{s}' = \{s_x, -|s_y|\}$, we can make a formal Fourier inversion of equation (5.10a) [11] and obtain the following explicit operator expression for the specific intensity:

$$I(\boldsymbol{x}, \boldsymbol{s}) = \frac{1}{2}\rho g v_g s_y [1 + \nabla^2/(4k^2)] \frac{k}{2\pi} \int_{-\infty}^{\infty} \exp(-\mathrm{i}\sqrt{k^2+\nabla^2/4}s_x \cdot \xi_x) W(\boldsymbol{x}, \xi_x \boldsymbol{e}_x)\, \mathrm{d}\xi_x \tag{5.11}$$

where $\boldsymbol{s} = \{s_x, s_y\}$, $s_y = \sqrt{1-s_x^2}$, and $\boldsymbol{e}_x$ is the unit vector in the x direction. Choosing the y-axis along $\boldsymbol{s}$, we see that this expression reduces to

$$I(\boldsymbol{x}, \boldsymbol{s}) = \frac{1}{2}\rho g v_g [1 + \nabla^2/(4k^2)] \frac{k}{2\pi} \int \delta(\boldsymbol{s}\cdot\boldsymbol{\xi}) W(\boldsymbol{x}, \boldsymbol{\xi})\, \mathrm{d}^2\xi \tag{5.12}$$

where $\mathrm{d}^2\xi = \mathrm{d}\xi_x\, \mathrm{d}\xi_y$ and $\delta(\boldsymbol{s}\cdot\boldsymbol{\xi})$ implies that the integral only has a contribution along the line $\boldsymbol{s}\cdot\boldsymbol{\xi} = 0$. If $W(\boldsymbol{x}, \boldsymbol{\xi})$ is a 'slow' function of x on the scale of $1/k = \lambda/(2\pi)$, equation (5.12) reduces to

$$I(\boldsymbol{x}, \boldsymbol{s}) = \frac{1}{2}\rho g v_g \frac{k}{2\pi} \int \delta(\boldsymbol{s}\cdot\boldsymbol{\xi}) W(\boldsymbol{x}, \boldsymbol{\xi})\, \mathrm{d}^2\xi. \tag{5.13}$$

Equations (5.12) and (5.13) only apply if we have no contribution from the opposite ray direction, i.e. if $I(\boldsymbol{x}, -\boldsymbol{s}) = 0$. In the general case, an explicit expression of the generalized specific intensity is obtained from the representation [11]

$$W(\boldsymbol{x}, \boldsymbol{\xi}) = \left(\frac{1}{2\pi}\right)^2 \int \mathcal{W}(\boldsymbol{x}, \boldsymbol{\kappa}) \exp(\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\xi})\, \mathrm{d}^2\kappa \tag{5.14}$$

where $\mathrm{d}^2\kappa = \mathrm{d}\kappa_x\, \mathrm{d}\kappa_y$ and

$$\mathcal{W}(\boldsymbol{x}, \boldsymbol{\kappa}) = \int W(\boldsymbol{x}, \boldsymbol{\xi}) \exp(-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\xi})\, \mathrm{d}^2\xi \tag{5.15}$$

is the Wigner distribution for the cross-spectral density function. Substituting equation (5.14) in equation (4.20) and introducing polar coordinates, we obtain

$$\begin{aligned} F(x, \omega) &= \frac{1}{2}\rho g v_g \left(\frac{1}{2\pi}\right)^2 \int \mathcal{W}(x, \kappa)(\kappa/k)\, \mathrm{d}^2\kappa \\ &= \int_0^{2\pi} \left[\frac{1}{2}\rho g v_g \left(\frac{1}{2\pi}\right)^2 \int_0^\infty \mathcal{W}(x, \kappa s)(\kappa^2/k)\, \mathrm{d}\kappa\right] s\, \mathrm{d}\theta(s) \end{aligned} \tag{5.16}$$

which, by comparison with equation (5.5), gives the general expression:

$$I(x, s) = \frac{1}{2}\rho g v_g \left(\frac{1}{2\pi}\right)^2 \int_0^\infty \mathcal{W}(x, \kappa s)(\kappa^2/k)\, \mathrm{d}\kappa. \tag{5.17}$$

In [11], equivalent inversion formulae for three-dimensional acoustical waves are derived and shown to reproduce the correct wave description of radiative energy transfer. The only difference between the description in [11] and the present one is that here we consider dispersive, two-dimensional surface waves. In particular, it was shown by examples in [11] that the exact, geometrical theory provides both the ordinary geometrical optics approximation to radiative energy transfer and a ray description of interfering fields. These examples apply directly to the present case as well.

From the representation in equation (5.1), we now easily find

$$\begin{aligned} I(x, s) = \frac{1}{2}\rho g v_g \frac{k}{2\pi} \int_0^{2\pi} \int_0^{2\pi} C(s_1, s_2) \frac{1}{2}|s_1 + s_2| \\ \times \delta\left(s - \frac{s_1 + s_2}{|s_1 + s_2|}\right) \exp[\mathrm{i}k(s_2 - s_1)\cdot x]\, \mathrm{d}\theta(s_1)\, \mathrm{d}\theta(s_2) \end{aligned} \tag{5.18}$$

where $\delta(s - s_0)$ is a short-hand notation for $\delta(\theta - \theta_0)$ with $s = \{\cos\theta, \sin\theta\}$ and $s_0 = \{\cos\theta_0, \sin\theta_0\}$. This expression is easily transformed to the one in equation (5.6). As in [11] we can now easily recover equation (5.1) by substituting equation (5.18) in equation (5.10*a*) and replacing ∇ in the operator expression by $\mathrm{i}k(s_2 - s_1)$. We see from equation (5.18) that we have a positive contribution from the plane wave component in the direction $s_1 = s_2 = s$. In addition, we have contributions from all pairs of correlated plane waves with interference fringes along s. The latter interference contributions are not necessarily positive.

The present exact, geometrical description of the radiative energy transfer is formally very similar to the classical description. But, although equations (5.5), (5.7), and (5.9) are formally identical to the corresponding classical relations, the generalized specific intensity $I(x, s)$ cannot generally be identified with the classical specific intensity $I_0(x, s)$. The reason for this is that the present geometrical description applies for any field that can be represented by equation (5.1) so that the theory includes proper wave phenomena like interference and diffraction which are excluded from the classical description. Therefore the generalized specific intensity must possess properties different from those of the classical specific intensity, even when the approximation in equation (5.9) is valid. In particular, the generalized specific intensity cannot be assumed to be a positive quantity (cf equation (2.6)) if interference and diffraction phenomena are to be included in the description. Nevertheless, the close formal analogy between the exact description derived here and the classical description of section 2 indicates that the latter can be derived by imposing appropriate approximations on the exact description. As we will see, this is the case in the quasihomogeneous approximation.

6. The quasihomogeneous approximation

We here use the same approximations as in [11] to derive the classical description of radiative energy transfer on the basis of the exact description of the preceding section.

The exact theory of the previous section applies for arbitrary fields that can be represented by the plane-wave expansion in equation (5.1), and it therefore also includes interfering and diffracted fields. To derive the classical theory we assume that the field is such that the directional correlation vanishes, i.e. $C(\mathbf{s}_1, \mathbf{s}_2)=0$, for $|\mathbf{s}_2-\mathbf{s}_1| \geq 2\sin(\theta_c/2)$ where the coherence angle θ_c is a measure of the largest angular difference between correlated plane waves of the field. If the coherence angle is small, we only have contributions to equation (5.1) when $\mathbf{s}_1 \approx \mathbf{s}_2$, i.e. when $q=|\mathbf{s}_2-\mathbf{s}_1| \ll 1$. Then $Q=\sqrt{1-q^2/4} \approx 1$ and equations (5.3) and (5.6) can, respectively, be written as

$$W(\mathbf{x}, \boldsymbol{\xi}) = \frac{2}{\rho g v_g} \int_0^{2\pi} I(\mathbf{x}, \mathbf{s}) \exp(\mathrm{i}k\mathbf{s} \cdot \boldsymbol{\xi})\, \mathrm{d}\theta(\mathbf{s}) \tag{6.1}$$

and

$$I(\mathbf{x}, \mathbf{s}) = \frac{1}{2} \rho g v_g \frac{k}{2\pi} \int S(\mathbf{s}, \mathbf{q}) \delta(\mathbf{q} \cdot \mathbf{s}) \exp(\mathrm{i}k\mathbf{q} \cdot \mathbf{x})\, \mathrm{d}^2 q. \tag{6.2}$$

Equation (6.1) can also be obtained by retaining only the leading term of the series development in equation (5.10*b*). This approximate solution of equations (3.21*a*, *b*) was first given by Dolin [23] by a totally different approach.

If the generalized specific intensity $I(\mathbf{x}, \mathbf{s})$ is a 'slow' function of $\mathbf{s}$ compared to the coherence angle, we can approximate $I(\mathbf{x}, \mathbf{s})$ by its local average over an angle $\Delta\theta$ much larger than the coherence angle. From equation (5.18) we then obtain

$$\begin{aligned} I(\mathbf{x}, \mathbf{s}) &\approx \frac{1}{\Delta\theta} \int_{\Delta\theta} I(\mathbf{x}, \mathbf{s})\, \mathrm{d}\theta(\mathbf{s}) \\ &= \frac{1}{\Delta\theta} \frac{1}{2} \rho g v_g \frac{k}{2\pi} \int\int_{\{(\mathbf{s}_1+\mathbf{s}_2)/|\mathbf{s}_1+\mathbf{s}_2| \in \Delta\theta\}} C(\mathbf{s}_1, \mathbf{s}_2) \\ &\quad \times \frac{1}{2} |\mathbf{s}_1+\mathbf{s}_2| \exp[\mathrm{i}k(\mathbf{s}_2-\mathbf{s}_1) \cdot \mathbf{x}]\, \mathrm{d}\theta(\mathbf{s}_1)\, \mathrm{d}\theta(\mathbf{s}_2) \\ &\approx \frac{1}{\Delta\theta} \frac{1}{2} \rho g v_g \frac{k}{2\pi} \int_{\Delta\theta} \int_{\Delta\theta} C(\mathbf{s}_1, \mathbf{s}_2) \exp[\mathrm{i}k(\mathbf{s}_2-\mathbf{s}_1) \cdot \mathbf{x}]\, \mathrm{d}\theta(\mathbf{s}_1)\, \mathrm{d}\theta(\mathbf{s}_2) \end{aligned} \tag{6.3}$$

where the last approximation applies because we only have contributions when $\mathbf{s}_1$ and $\mathbf{s}_2$ are within the coherence angle, i.e. when $|\mathbf{s}_2-\mathbf{s}_1| \ll \Delta\theta$ so that $\mathbf{s}_1 \approx \mathbf{s}_2$. In this case, we see that the specific intensity is positive since it is proportional to the intensity of the field from an angular aperture $\Delta\theta$. From equations (6.1) and (6.2) it can now be shown that the generalized specific intensity must also be a 'slow' function of $\mathbf{x}$ compared to the local transverse coherence length, which is a measure of the maximum distance $\xi=|\mathbf{x}_2-\mathbf{x}_1|$ for which $W(\mathbf{x}_1, \mathbf{x}_2, \omega) \neq 0$ along the line $\mathbf{x} \cdot \mathbf{s} = \text{constant}$ [9, 11]. Then equations (5.9) and (5.13) also apply and all the radiative transfer relations are formally equal to the classical ones.

In this approximation, our generalized specific intensity possesses *all* the properties of the classical specific intensity and, thus, can be *identified* with the latter [9, 11], i.e.

$$I(\mathbf{x}, \mathbf{s}) = I_0(\mathbf{x}, \mathbf{s}). \tag{6.4}$$

Substituting from equation (6.4) in equation (6.1), we obtain

$$W(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{2}{\rho g v_{\rm g}} \int_0^{2\pi} I_0(\boldsymbol{x}, \boldsymbol{s}) \exp(\mathrm{i}k\boldsymbol{s} \cdot \boldsymbol{\xi})\, \mathrm{d}\theta(\boldsymbol{s}). \tag{6.5}$$

Equation (6.5) is a generalized van Cittert–Zernike theorem which defines our quasihomogeneous field model. This model applies for fields for which the radiative energy transfer is described by the classical theory. The classical van Cittert–Zernike theorem, which applies for radiation from incoherent sources, reduces the calculation of the cross-spectral density function to calculation of the diffracted field around a focus in an equivalent diffraction problem. Similarly, the generalized theorem in equation (6.5) corresponds to the equivalent diffraction problem of calculating the diffracted field around a focus in the so-called modified Debye approximation [12–14].

The present quasihomogeneous model is formally different from the one due to Carter and Wolf [24], but the physical approximations involved are closely related. A completely homogeneous field is one for which the cross-spectral density and the specific intensity are independent of the position $\boldsymbol{x}$ whereas a *quasihomogeneous* field is one for which the $\boldsymbol{x}$-dependence of these quantities are 'slow' over distances comparable to a local transverse coherence length of the field. However, in contrast to the Carter–Wolf model, our model does not imply that the dependencies on the mean and difference coordinates are separable.

Within the quasihomogeneous approximation, the inversion formulae (equations (5.13) and (5.16)) can be replaced by simple Fourier inversion formulae. If the bundle of rays contributing to equation (6.5) subtends an angle less than π so that all ray directions $\boldsymbol{s}$ have a positive s_y component, we obtain by direct Fourier inversion of equation (6.5)

$$I_0(\boldsymbol{x}, \boldsymbol{s}) = \frac{1}{2} \rho g v_{\rm g} s_y \frac{k}{2\pi} \int_{-\infty}^{\infty} W(\boldsymbol{x}, \xi_x \boldsymbol{e}_x) \exp(-\mathrm{i}k s_x \xi_x)\, \mathrm{d}\xi_x. \tag{6.6}$$

Here, $\boldsymbol{x} = \{x, y\}$, $\boldsymbol{s} = \{s_x, s_y\}$ is a unit vector, and $\boldsymbol{e}_x$ is the unit vector in the x direction. Equation (6.6) is equivalent to Walther's first definition of the radiance function [25]. For $\boldsymbol{s}$ chosen along the y axis, this expression reduces to equation (5.13).

If we have directional ambiguity we must use a modified inversion formula in order to distinguish between rays in the directions $\boldsymbol{s} = \{s_x, |s_y|\}$ and $\boldsymbol{s}' = \{s_x, -|s_y|\}$. This can be achieved in several ways, for example by using the general inversion formula in equations (5.15) and (5.17). A possible Fourier inversion formula for this case is given by

$$I_0(\boldsymbol{x}, \boldsymbol{s}) = \frac{1}{2} \rho g v_{\rm g} s_y \frac{k}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2} \left[W(\boldsymbol{x}, \boldsymbol{\xi}) + \frac{1}{\mathrm{i}k s_y} \frac{\partial}{\partial \xi_y} W(\boldsymbol{x}, \boldsymbol{\xi}) \right]_{\boldsymbol{\xi} = \xi_x \boldsymbol{e}_x} \exp(-\mathrm{i}k s_x \xi_x)\, \mathrm{d}\xi_x. \tag{6.7}$$

By choosing the y axis along $\boldsymbol{s}$ we then obtain

$$I_0(\boldsymbol{x}, \boldsymbol{s}) = \frac{1}{2} \rho g v_{\rm g} \frac{k}{2\pi} \int \frac{1}{2} \left[W(\boldsymbol{x}, \boldsymbol{\xi}) + \frac{1}{\mathrm{i}k} \boldsymbol{s} \cdot \nabla_{\xi} W(\boldsymbol{x}, \boldsymbol{\xi}) \right] \delta(\boldsymbol{s} \cdot \boldsymbol{\xi})\, \mathrm{d}^2\xi. \tag{6.8}$$

In [9] and [11], it was shown that the present quasihomogeneous approximation implies that the field obeys Gaussian statistics. The statistics of such Gaussian fields are completely determined by the second-order coherence functions which, through equation (6.5) and equations (3.18)–(3.20), are now given by the classical specific intensity. Thus a *complete* statistical wave model is provided in terms of the classical description of radiative energy transfer. This statistical wave model, defined in terms of the phenomenological variables, constitutes the rational justification of the phenomenological description in complete analogy with the statistical justification of thermodynamics [26].

7. Summary and discussion

In this paper, we have shown that the energy transfer for partially coherent, linear surface gravity waves in water of constant depth can be described by the classical theory of radiative energy transfer if the wavefield can be approximated by a quasihomogeneous wave model. In doing so, we have first developed a coherence theory description for linear water waves and shown that the partially coherent wave motion is described by the same reduced Wolf equations that are used in optical coherence theory. On that basis, we have derived exact average relations for the wave energy transfer and shown them to be completely analogous to the corresponding relations for non-dispersive acoustical waves. Using previous results for non-dispersive waves, we have then established an exact geometrical description of radiative energy transfer which is formally very similar to the classical theory. Finally, we have introduced the quasihomogeneous wave model for which the exact description reduces to the classical theory of radiative energy transfer.

Our results illustrate the basic unity of wave phenomena that are physically different. We have found that both the propagation equations and the energy relations are given by almost identical formulae for non-dispersive acoustical waves and for dispersive water waves. The only difference is due to the dispersion which, for water waves, causes the group velocity to differ from the phase velocity. Our results also demonstrate that a spectral description is very powerful for revealing analogies between different wave phenomena.

Geometrical ray-trace calculations of wave energy transfer are routinely used in fields like harbour engineering and wave energy utilization for predicting the effects of near-shore refraction and focusing on the local wave climate [3, 18, 19, 27]. Our results indicate that such an approach is justified if the quasihomogeneous wave model is a good approximation of the sea state, i.e. if the specific intensity is a 'slow' function of direction and position compared to, respectively, the coherence angle and the local transverse coherence length of the field. In addition, the exact energy relations derived in section 5 can be applied in regions where the quasihomogeneous approximation breaks down and interference and diffraction phenomena becomes appreciable. However, the present results are limited to linear waves in water of constant depth. Further work is needed to extend the description to include depth variations and nonlinearities.

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